Using Sum, Difference, and Double-Angle Identities

Sum and Difference Identities

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Simplifying Trigonometric Expressions

1. Express $\cos 100^0 \cos 80^0 + \sin 80^0 \sin 100^0$ as a trig function of a single angle.

This function has the same pattern as $\cos(A - B)$, with $A = 100^0$ and $B = 80^0$.

$$\cos 100 \cos 80 + \sin 80 \sin 100 = \cos(100^0 - 80^0)$$
$$= \cos 20^0$$

2. Express $\sin\frac{\pi}{3}\cos\frac{\pi}{6} - \cos\frac{\pi}{3}\sin\frac{\pi}{6}$ as a single trig function.

This function has the same pattern as $\sin(A - B)$, with

$A = \frac{\pi}{3}$ and $B = \frac{\pi}{6}$.

$$\sin\frac{\pi}{3}\cos\frac{\pi}{6} - \cos\frac{\pi}{3}\sin\frac{\pi}{6} = \sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$
$$= \sin\frac{\pi}{6}$$

Finding Exact Values

1. Find the exact value for sin 75^0.

Think of the angle measures that produce exact values: 30^0, 45^0, and 60^0.

Use the sum and difference identities.

Which angles, used in combination of addition or subtraction, would give a result of 75^0?

$$\sin 75^0 = \sin(30^0 + 45^0)$$

$$= \sin 30^0 \cos 45^0 + \cos 30^0 \sin 45^0$$

$$= \left(\frac{1}{2} \times \frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{2} + \sqrt{6}}{4}$$

Finding Exact Values

2. Find the exact value for cos 15^0.

$$\cos 15^0 = \cos(45^0 - 30^0)$$
$$= \cos 45^0 \cos 30^0 + \sin 45^0 \sin 30^0$$
$$= \left(\frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2} \times \frac{1}{2}\right)$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

3. Find the exact value for $\sin\frac{5\pi}{12}$.

$$\sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$
$$= \sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$
$$= \left(\frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2} \times \frac{1}{2}\right)$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\frac{\pi}{3} = \frac{4\pi}{12}$$
$$\frac{\pi}{4} = \frac{3\pi}{12}$$
$$\frac{\pi}{6} = \frac{2\pi}{12}$$

Using the Sum and Difference Identities

Prove $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta.$

$\cos\left(\frac{\pi}{2}+\theta\right)$	$=$	$-\sin\theta.$
$= \cos\frac{\pi}{2}\cos\theta - \sin\frac{\pi}{2}\sin\theta$		$-\sin\theta$
$= (0)(\cos\theta) - (1)(\sin\theta)$		
$= -(1)(\sin\theta)$		
$= -\sin\theta$		

L.S. = R.S.

Using the Sum and Difference Identities

Given $\cos\theta = \frac{3}{5}$, where $0 \le \theta \le \frac{\pi}{2}$,

find the exact value of $\cos(\theta + \frac{\pi}{6})$.

$$\cos\theta = \frac{x}{r}$$

$x = 3$

$r = 5$

$r^2 = x^2 + y^2$

$y^2 = r^2 - x^2$

$= 5^2 - 3^2$

$= 16$

$y = \pm 4$

$$\cos(\theta + \frac{\pi}{6}) = \cos\theta\cos\frac{\pi}{6} - \sin\theta\sin\frac{\pi}{6}$$

$$= (\frac{3}{5})(\frac{\sqrt{3}}{2}) - (\frac{4}{5})(\frac{1}{2})$$

$$= \frac{3\sqrt{3} - 4}{10}$$

Therefore, $\cos(\theta + \frac{\pi}{6}) = \frac{3\sqrt{3} - 4}{10}$.

Using the Sum and Difference Identities

$$\text{Given } \sin A = \frac{2}{3} \text{ and } \cos B = \frac{4}{5},$$

where A and B are acute angles,

find the exact value of $\sin(A + B)$.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$= \left(\frac{2}{3}\right)\left(\frac{4}{5}\right) + \left(\frac{\sqrt{5}}{3}\right)\left(\frac{3}{5}\right)$$

	A	B
x	$\sqrt{5}$	4
y	2	3
r	3	5

$$= \frac{8 + 3\sqrt{5}}{15}$$

$$\text{Therefore, } \sin(A+B) = \frac{8+3\sqrt{5}}{15}.$$

Double-Angle Identities

The identities for the sine and cosine of the sum of two numbers can be used, when the two numbers A and B are equal, to develop the identities for sin 2A and cos 2A.

$$\sin 2A = \sin(A + A)$$
$$= \sin A \cos A + \cos A \sin A$$
$$= 2 \sin A \cos A$$

$$\cos 2A = \cos(A + A)$$
$$= \cos A \cos A - \sin A \sin A$$
$$= \cos^2 A - \sin^2 A$$

Identities for sin 2x and cos 2x:

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos 2x = 2\cos^2 x - 1$$
$$\cos 2x = 1 - 2\sin^2 x$$

Double-Angle Identities

Express each in terms of a single trig function.

a) $2 \sin 0.45 \cos 0.45$

$$\sin 2x = 2\sin x \cos x$$

b) $\cos^2 5 - \sin^2 5$

$$\cos 2x = \cos^2 x - \sin^2 x$$

A group of Friars opened a florist shop to help with their belfry payments. Everyone liked to buy flowers from the Men of God, so their business flourished. A rival florist became upset that his business was suffering because people felt compelled to buy from the Friars, so he asked the Friars to cut back hours or close down. The Friars refused. The florist went to them and begged that they shut down.

Double-Angle Identities

Verify the identity $\tan A = \dfrac{1-\cos 2A}{\sin 2A}$.

$\tan A$	
	$= \dfrac{1-(\cos^2 A - \sin^2 A)}{2\sin A\cos A}$
	$= \dfrac{1-\cos^2 A + \sin^2 A)}{2\sin A\cos A}$
	$= \dfrac{\sin^2 A + \sin^2 A}{2\sin A\cos A}$
	$= \dfrac{2\sin^2 A}{2\sin A\cos A}$
	$= \dfrac{\sin A}{\cos A}$
	$= \tan A$

L.S = R.S.

Double-Angle Identities

Verify the identity $\tan x = \dfrac{\sin 2x}{1+\cos 2x}$.

$\tan x$	$= \dfrac{2\sin x \cos x}{1+2\cos^2 x - 1}$
	$= \dfrac{2\sin x \cos x}{2\cos^2 x}$
	$= \dfrac{\sin x}{\cos x}$
	$= \tan x$

L.S = R.S.

Double-Angle Equations

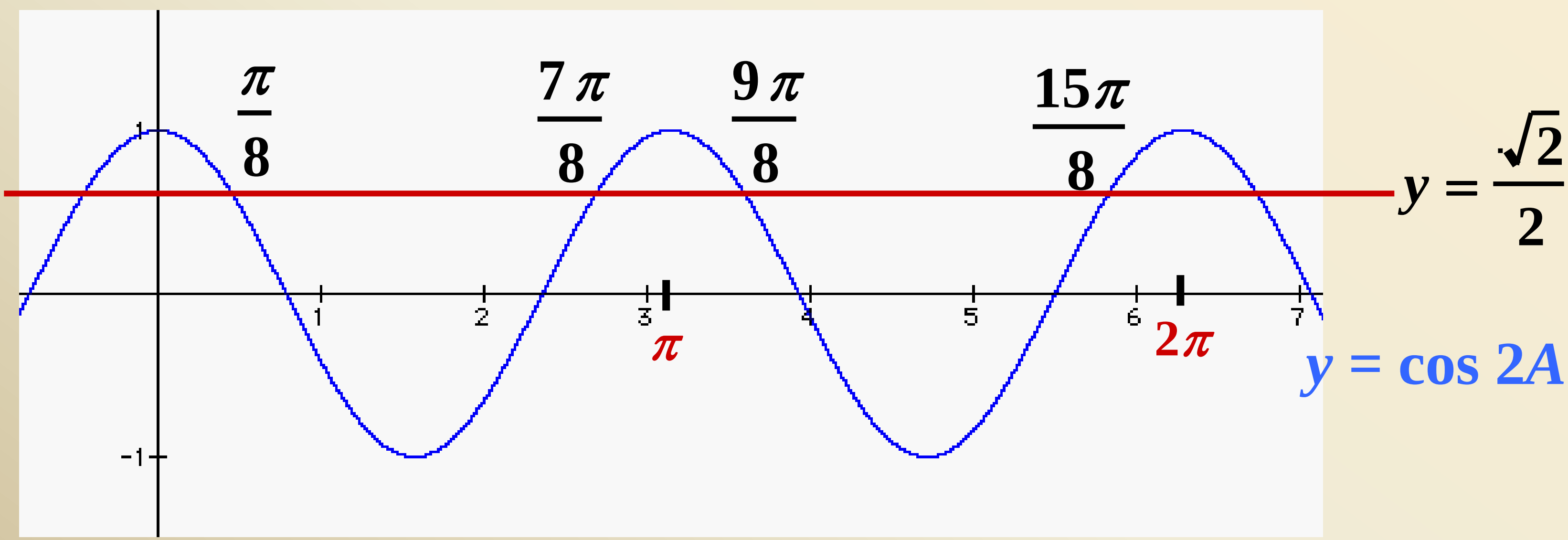
Find A given $\cos 2A = \frac{\sqrt{2}}{2}$ where $0 \le A \le 2\pi$.

For $\cos A = \frac{\sqrt{2}}{2}$ where $0 \le A \le 2\pi$, $A = \frac{\pi}{4}$ and $\frac{7\pi}{4}$.

$$2A = \frac{\pi}{4} + 2\pi n, \frac{7\pi}{4} + 2\pi n.$$

$$2A = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$$

$$A = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$$



Double-Angle Equations

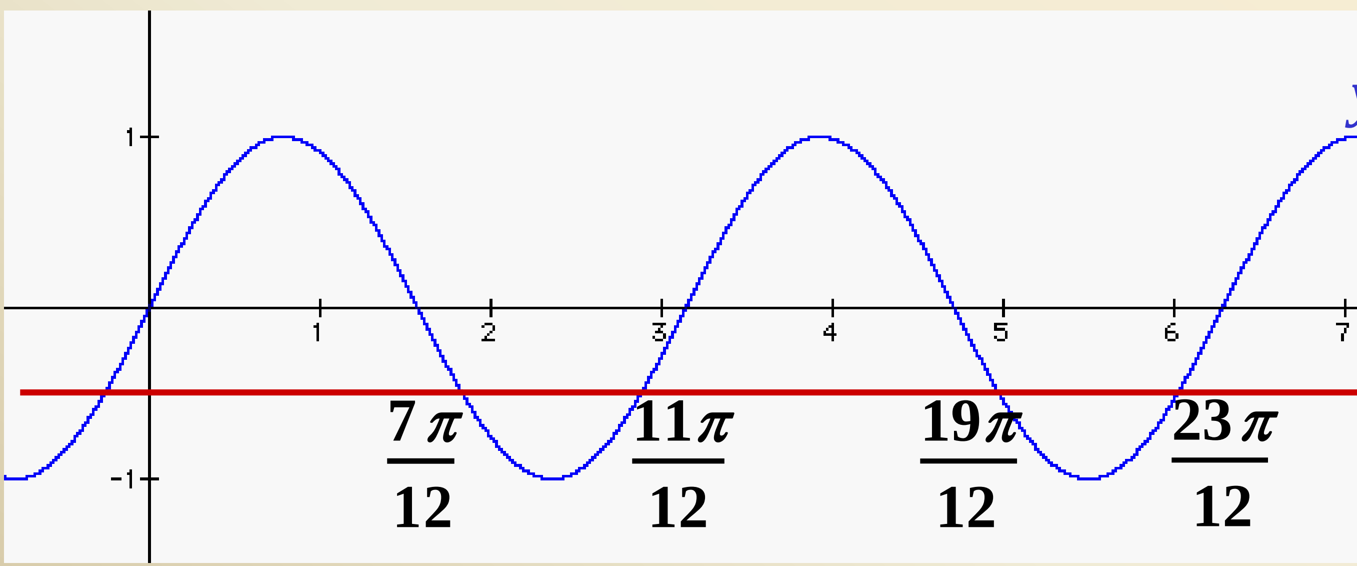
Find A given $\sin 2A = -\frac{1}{2}$ where $0 \le A \le 2\pi$.

For $\sin A = -\frac{1}{2}$ where $0 \le A \le 2\pi$, $A = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

$$2A = \frac{7\pi}{6} + 2\pi n, \frac{11\pi}{6} + 2\pi n.$$

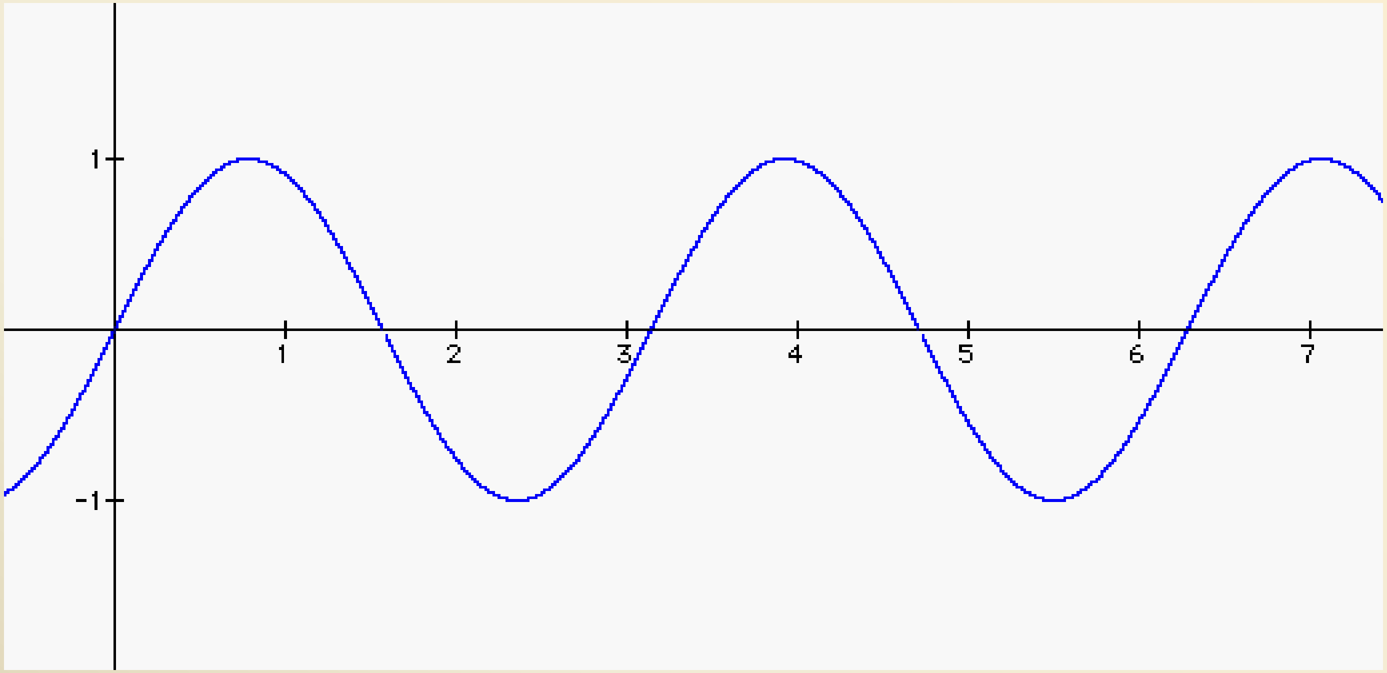
$$2A = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$$

$$A = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$



Using Technology

Graph the function $f(x) = \dfrac{2\tan x}{1+\tan^2 x}$ and predict the period.



The period is π.

Rewrite $f(x)$ as a single trig function: $f(x) = \sin 2x$

Identities

Prove $\dfrac{2\tan x}{1+\tan^2 x} = \sin 2x$.

L.S.	R.S.
$\dfrac{\frac{2\sin x}{\cos x}}{\sec^2 x}$	$= 2\sin x\cos x$
$= \dfrac{\frac{2\sin x}{\cos x}}{\frac{1}{\cos^2 x}}$	
$= \dfrac{2\sin x}{\cos x} \times \dfrac{\cos^2 x}{1}$	
$= 2\sin x\cos x$	

L.S. = R.S.

Applying Skills to Solve a Problem

The horizontal distance that a soccer ball will travel, when kicked at an angle θ, is given by $d = \frac{2v_0^2}{g}\sin\theta\cos\theta$,

where d is the horizontal distance in metres, v_0 is the initial velocity in metres per second, and g is the acceleration due to gravity, which is 9.81 m/s^2.

a) Rewrite the expression as a sine function.

Use the identity $\sin 2A = 2\sin A \cos A$:

$$d = \frac{2v_0^2}{g}\sin\theta\cos\theta$$

$$d = \frac{v_0^2}{g}\sin 2\theta$$

Applying Skills to Solve a Problem [cont'd]

b) Find the distance when the initial velocity is 20 m/s.

$$d = \frac{v_0^2}{g}\sin 2\theta$$

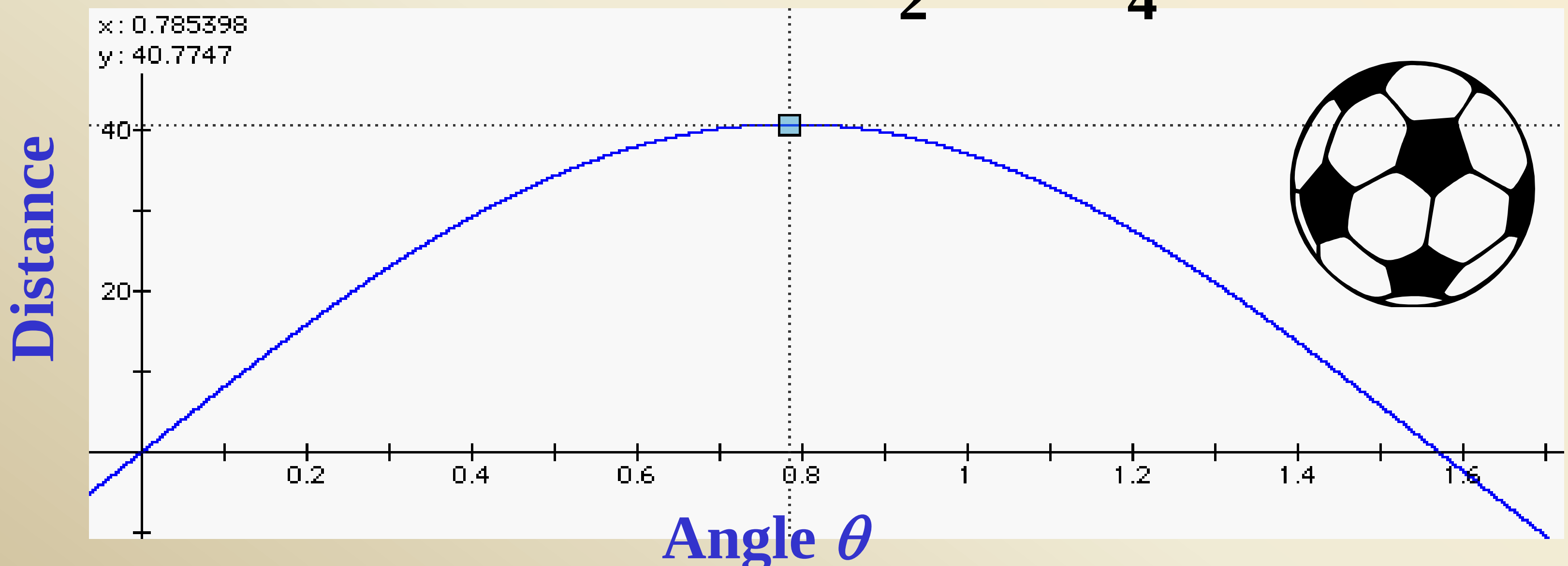
$$d = \frac{(20)^2}{9.81}\sin 2\theta$$

From the graph, the maximum distance occurs when $\theta = 0.785398$.

The maximum distance is 40.7747 m.

The graph of sin θ reaches its maximum when $\theta = \frac{\pi}{2}$.

Sin 2θ will reach a maximum when $2\theta = \frac{\pi}{2}$ or $\theta = \frac{\pi}{4}$.



Assignment

Again they refused. So the florist then hired Hugh McTaggert, the biggest meanest thug in town. He went to the Friars' shop beat them up, destroyed their flowers, trashed their shop, and said that if they didn't close, he'd be back. Well, totally terrified, the Friars closed up shop and hid in their rooms. This proved that Hugh, and only Hugh, can prevent florist friars.