Polynomial Functions and Equations Lesson #7:
Investigating the Graphs of Polynomial Functions - Part One

Review of Zeros, Roots, and x-intercepts

Fill in the blanks in the following statement regarding the function with equation $y = P(x)$.

"The zeros of the function, the x-intercepts of the graph of the function, and the roots of the corresponding equation $y = 0$ are the same numbers."

Unique Factorization Theorem

This theorem states that every polynomial function of degree $n \geq 1$ can be written as the product of a leading coefficient, c, and n linear factors to get

$$P(x) = c(x - a_1)(x - a_2)(x - a_3)...(x - a_n)$$

This theorem implies two ***important*** points for polynomial functions of degree $n \geq 1$:

Point #1: Every polynomial function can be written as a product of its factors and a leading coefficient.

Point #2: Every polynomial function has the same number of factors as its degree. The factors may be real or complex, and may be repeated.

In this lesson we will consider only polynomial functions where the leading coefficient, c, is either 1 or −1. In lesson 10, we will consider polynomial functions with a leading coefficient other than ±1.

The graph of $P(x) = x^3 - 2x^2 - 5x + 6$ is shown. The polynomial has integral zeros.

a) Use the graph to

i) state the zeros of the polynomial

$-2, 1, 3$

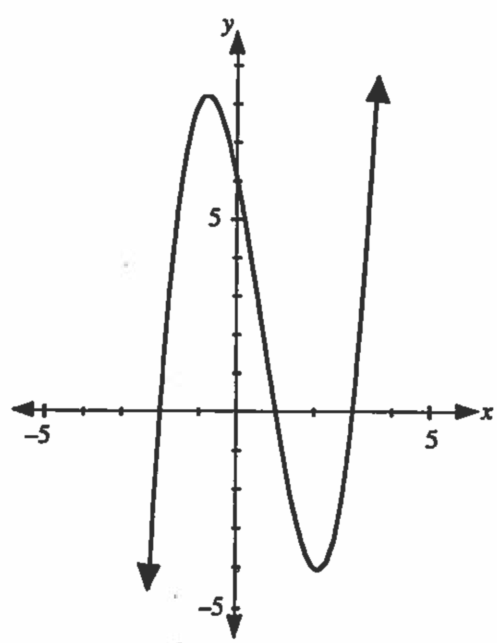
ii) state the factors of the polynomial

$x+2, \ x-1, \ x-3$

iii) write the polynomial in factored form

$P(x) = (x+2)(x-1)(x-3)$

b) Use a graphing calculator to sketch $P(x)$ in expanded form and in factored form to verify the above answers.



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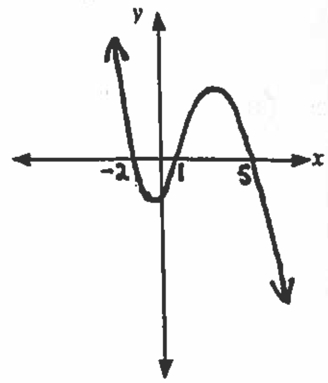
Class Ex. #2

a) Use a graphing calculator to sketch the graph of the polynomial function $P(x) = -x^3 + 4x^2 + 7x - 10$.



b) Use the graph to state the x-intercepts.

-2, 1, and 5

c) Write the polynomial in factored form.

$P(x) = -(x+2)(x-1)(x-5)$

d) Circle the correct alternative:

- The left arm of the graph is (rising / falling).
- The right arm of the graph is (rising / falling).
- The degree of the polynomial is (even / odd).
- The leading coefficient of the polynomial is (positive / negative).

Note: The investigative assignment in this lesson will develop the relationships between the directions of the arms of the graph of a polynomial, the degree of the polynomial, and the sign of the leading coefficient of the polynomial.

Complete Assignment Questions #1 - #3

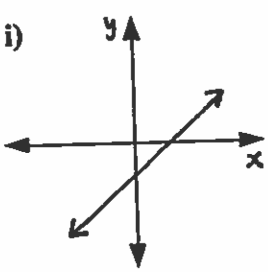
Assignment

1. In each question use a graphing calculator to:

i) sketch the graph of the polynomial function
ii) state the zeros of the polynomial function
iii) write the polynomial function in factored form.

a) $P(x) = x - 2$

i)

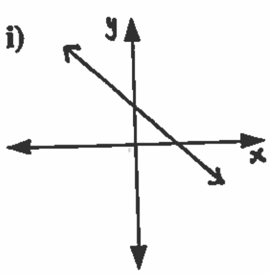


ii) 2

iii) $P(x) = (x - 2)$

b) $P(x) = -x + 2$

i)

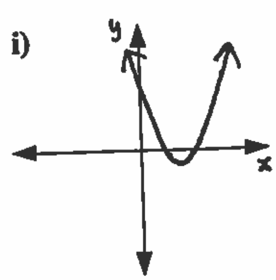


ii) 2

iii) $P(x) = -(x - 2)$

c) $P(x) = x^2 - 6x + 8$

i)

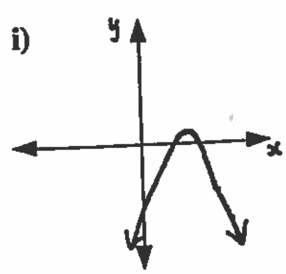


ii) $2, 4$

iii) $P(x) = (x-2)(x-4)$

d) $P(x) = -x^2 + 6x - 8$

i)



ii) $2, 4$

iii) $P(x) = -(x-2)(x-4)$

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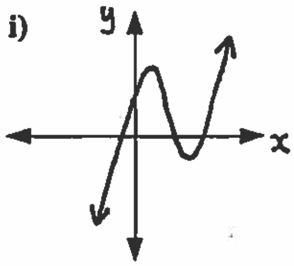
e) $P(x) = x^3 - 7x^2 + 7x + 15$

i)

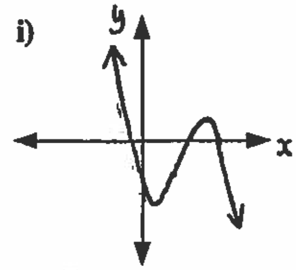


ii) $-1, 3, 5$

iii) $P(x) = (x+1)(x-3)(x-5)$

f) $P(x) = -x^3 + 7x^2 - 7x - 15$

i)

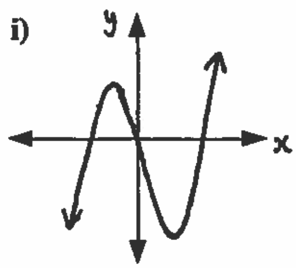


ii) $-1, 3, 5$

iii) $P(x) = -(x+1)(x-3)(x-5)$

g) $P(x) = x^3 - x^2 - 12x$

i)

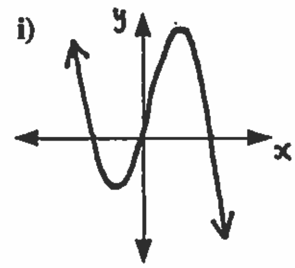


ii) $-3, 0, 4$

iii) $P(x) = x(x+3)(x-4)$

h) $P(x) = -x^3 + x^2 + 12x$

i)

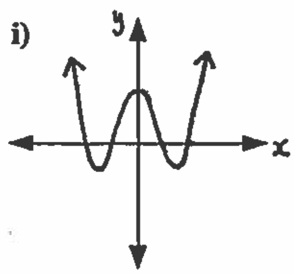


ii) $-3, 0, 4$

iii) $P(x) = -x(x+3)(x-4)$

i) $P(x) = x^4 - 5x^2 + 4$

i)

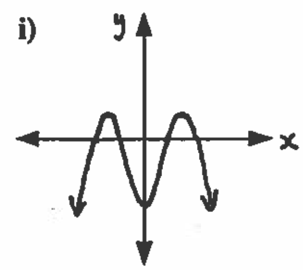


ii) $-2, -1, 1, 2$

iii) $P(x) = (x+2)(x+1)(x-1)(x-2)$

j) $P(x) = -x^4 + 5x^2 - 4$

i)



ii) $-2, -1, 1, 2$

iii) $P(x) = -(x+2)(x+1)(x-1)(x-2)$

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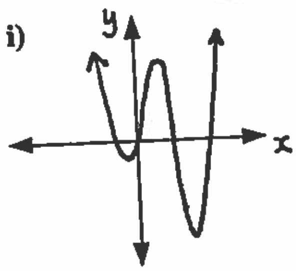
2. In each question use a graphing calculator to
   i) sketch the graph of the polynomial function
   ii) state the zeros of the polynomial function
   iii) write the polynomial function in factored form

a) $P(x) = x^4 - 7x^3 + 7x^2 + 15x$

i)

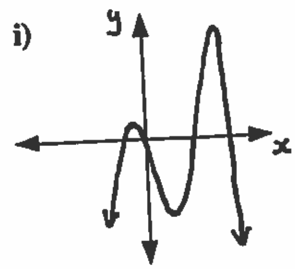


ii) $-1, 0, 3, 5$

iii) $P(x) = x(x+1)(x-3)(x-5)$

b) $P(x) = -x^4 + 7x^3 - 7x^2 - 15x$

i)

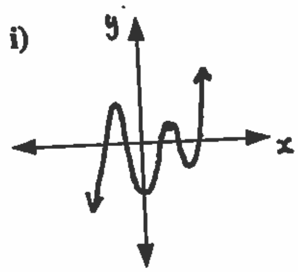


ii) $-1, 0, 3, 5$

iii) $P(x) = -x(x+1)(x-3)(x-5)$

c) $P(x) = x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12$

i)

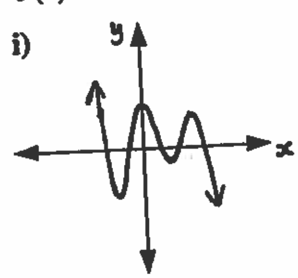


ii) $-2, -1, 1, 2, 3$

iii) $P(x) = (x+2)(x+1)(x-1)(x-2)(x-3)$

d) $P(x) = -x^5 + 3x^4 + 5x^3 - 15x^2 - 4x + 12$

i)

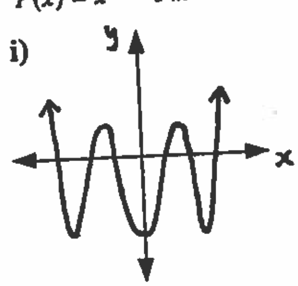


ii) $-2, -1, 1, 2, 3$

iii) $P(x) = -(x+2)(x+1)(x-1)(x-2)(x-3)$

e) $P(x) = x^6 - 14x^4 + 49x^2 - 36$

i)

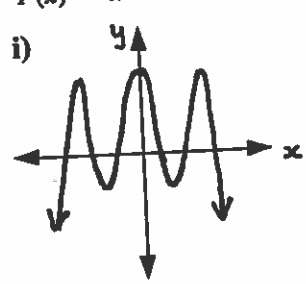


ii) $-3, -2, -1, 1, 2, 3$

iii) $P(x) = (x+3)(x+2)(x+1)(x-1)(x-2)(x-3)$

f) $P(x) = -x^6 + 14x^4 - 49x^2 + 36$

i)



ii) $-3, -2, -1, 1, 2, 3$

iii) $P(x) = -(x+3)(x+2)(x+1)(x-1)(x-2)(x-3)$

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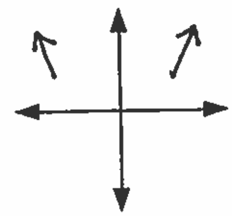
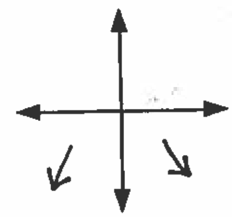
3. Based on your observations from questions #1 and #2, circle the correct choice in each of the following statements.

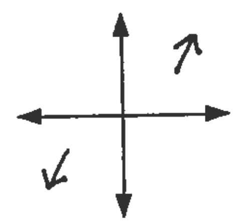
   a) If the graph of a polynomial has ***two rising*** arms, then the degree of the polynomial is (even, odd) and the leading coefficient is (positive, negative).



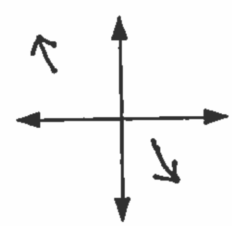
   b) If the graph of a polynomial has ***two falling arms***, then the degree of the polynomial is (even, odd) and the leading coefficient is (positive, negative).



   c) If the graph of a polynomial has a ***right*** arm ***rising*** **and** the ***left*** arm ***falling***, then the degree of the polynomial is (even, odd) and the leading coefficient is (positive, negative).



   d) If the graph of a polynomial has a ***right*** arm ***falling*** **and** the ***left*** arm ***rising***, then the degree of the polynomial is (even, odd) and the leading coefficient is (positive, negative).



   e) **The leading coefficient is positive** if the (left, right) arm is (rising, falling).

   f) **The leading coefficient is negative** if the (left, right) arm is (rising, falling).

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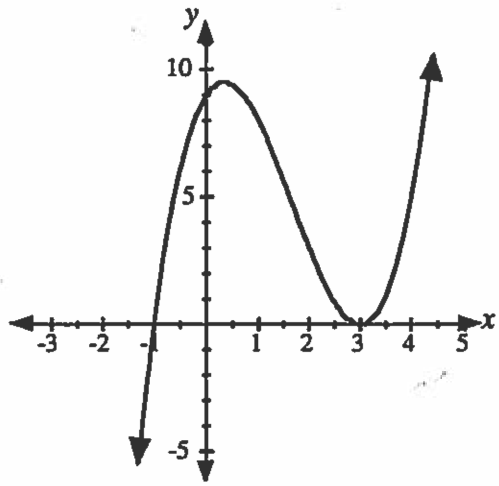
Polynomial Functions and Equations Lesson #8:
Investigating the Graphs of Polynomial Functions - Part Two

Repeated Factors

The graph of the polynomial function $P(x) = (x + 1)(x - 3)^2$ is shown.



The polynomial has two factors, one of which is repeated. This means that the function has **two distinct** zeros, one of which is a repeated zero.

The factors are $(x - 3)$ which is repeated, and $(x + 1)$.

The zeros of the function are therefore 3, which is repeated, and -1.

The graph of the function has two different x-intercepts, and we say that the function has **two real distinct zeros,** -1 and 3.

- The x-intercept at -1 represents a real zero of the function.
- The x-intercept of 3 represents **two real equal zeros** of the function.

The repeated zero of 3 is said to be a zero of **multiplicity** 2.
The zero of -1 is a zero of **multiplicity** 1.

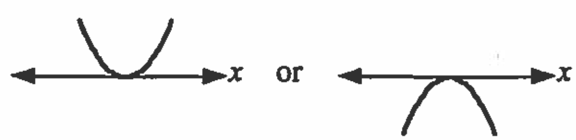
Multiplicity

The **multiplicity** of a zero corresponds to the number of times a factor is repeated in the function.

In this lesson, we will investigate how the multiplicity of a zero affects the shape of the graph of a polynomial function. In order to do this, we have to define the following terms.

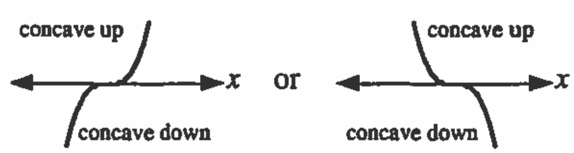
Tangent

A polynomial graph is **tangent** to the x-axis at a point where the graph **touches** the x-axis and does not cross through it.



Point of Inflection

A polynomial graph has a **point of inflection** on the x-axis if the graph **changes concavity** at a point on the x-axis.



In this lesson we will consider only polynomials where the leading coefficient, c, is either 1 or -1.

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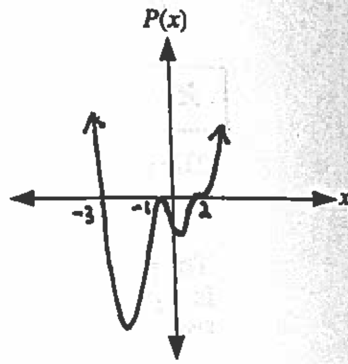
x. #1

Consider the polynomial function

$P(x) = x^6 - x^5 - 11x^4 + 13x^3 + 26x^2 - 20x - 24$

$= (x+3)(x+1)^2(x-2)^3$.



a) Sketch the graph of $P(x)$ using the window

x: [–5, 5, 1] y: [–100, 100, 20]

b) Complete the chart below to state the zeros of $P(x)$, their multiplicities, and whether each zero

- passes straight through the x-axis
- is tangent to the x-axis, or
- has a point of inflection.

zero	multiplicity	description
−3	1	passes through the x-axis
−1	2	is tangent to the x-axis
2	3	has a point of inflection

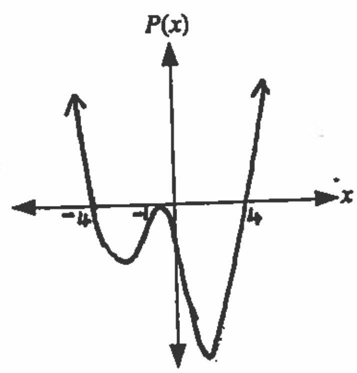
c) Complete the following.

- The degree of $P(x)$ is 6.
- The sum of the multiplicities of the zeros of $P(x)$ is 6.

Ex. #2

A polynomial function has the equation

$P(x) = x^4 + 2x^3 - 15x^2 - 32x - 16$.



a) Sketch the graph of $P(x)$ using the window

x: [–6, 6, 1] y: [–150, 100, 20]

b) Complete the chart below.

zero	multiplicity	description
−4	1	passes through the x-axis
−1	2	is tangent to the x-axis
4	1	passes through the x-axis

c) Complete the following.

- The degree of $P(x)$ is 4.
- The sum of the multiplicities of the zeros of $P(x)$ is 4.

d) Write the polynomial in the form $P(x) = (x-a)(x-b)(x-c)^2$.

$P(x) = (x+4)(x-4)(x+1)^2$

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Note

The investigative assignment in this lesson will develop the relationships between the multiplicities of the zeros of a polynomial function and its graph.

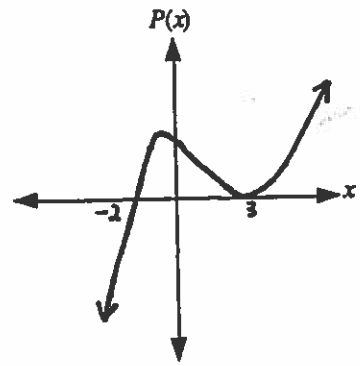
Complete Assignment Questions #1 - #15

Assignment

In this assignment, choose appropriate windows which will enable you to investigate all the characteristics of the functions.

1. a) Graph $P(x) = x^3 - 4x^2 - 3x + 18$ and complete the table.

zero	multiplicity
-2	1
3	2

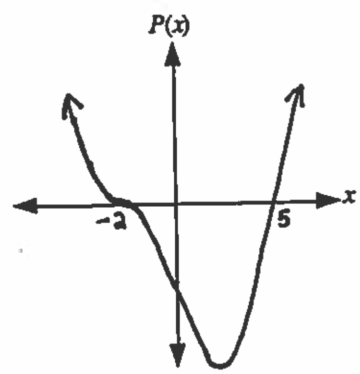


b) Write the polynomial in the form $P(x) = (x-a)(x-b)^2$, where $a, b \in I$.

$P(x) = (x+2)(x-3)^2$

2. a) Graph $P(x) = x^4 + x^3 - 18x^2 - 52x - 40$ and complete the table.

zero	multiplicity
-2	3
5	1

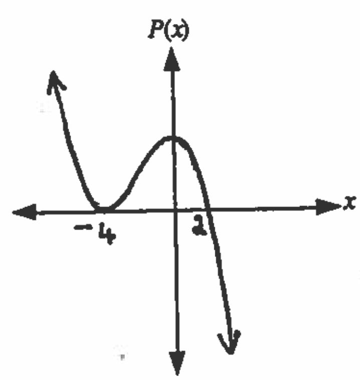


b) Write the polynomial function in factored form.

$P(x) = (x-5)(x+2)^3$

3. a) Graph $P(x) = -x^3 - 6x^2 + 32$ and complete the table.

zero	multiplicity
-4	2
2	1



b) Write the polynomial in the form $P(x) = -(x-a)(x-b)^2$, $a, b \in I$.

$P(x) = -(x-2)(x+4)^2$

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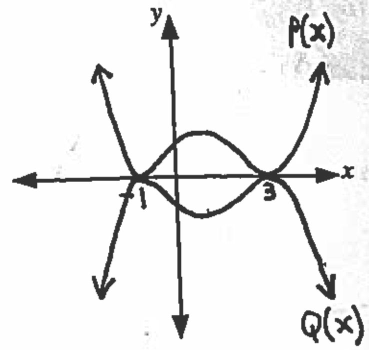
4\. a) Sketch the graphs of $P(x) = x^4 - 4x^3 - 2x^2 + 12x + 9$ and $Q(x) = -x^4 + 4x^3 + 2x^2 - 12x - 9$ on the grid.



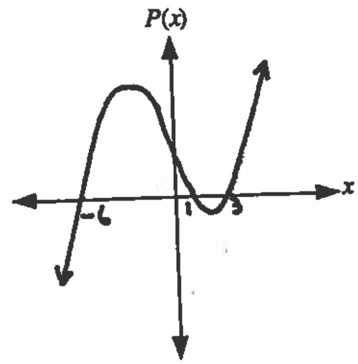
b) State the zeros, their multiplicities, and the y-intercept of each polynomial function.

	zero	multiplicity	y-intercept
P(x)	-1 3	2 2	9
Q(x)	-1 3	2 2	-9

c) Write the equations of the polynomials in factored form.

$P(x) = (x+1)^2(x-3)^2$ $\quad$ $Q(x) = -(x+1)^2(x-3)^2$

5\. A cubic polynomial function has the equation $P(x) = ax^3 + bx^2 + cx + d$ with a leading coefficient of 1. The zeros of the polynomial are −6, 1, and 3.



a) Sketch the graph of $P(x)$ and write the equation of the polynomial in factored form.

$P(x) = (x+6)(x-1)(x-3)$

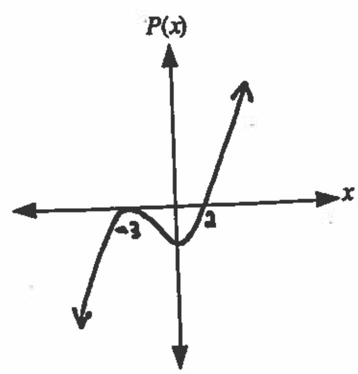
b) Determine the values of a, b, c, and d in $P(x)$.

$P(x) = (x+6)(x^2-4x+3)$

$P(x) = x^3 - 4x^2 + 3x + 6x^2 - 24x + 18$

$P(x) = x^3 + 2x^2 - 21x + 18$ $\quad$ $a = 1 \quad b = 2 \quad c = -21 \quad d = 18$

6\. A cubic polynomial function has the equation $P(x) = ax^3 + bx^2 + cx + d$ with a leading coefficient of 1. The function has two real distinct zeros. The zero 2 has multiplicity one, and the zero −3 has multiplicity two.



a) Sketch the graph of the function and write the equation of the polynomial in factored form.

$P(x) = (x-2)(x+3)^2$

b) Determine the values of a, b, c, and d in $P(x)$.

$P(x) = (x-2)(x^2+6x+9)$

$P(x) = x^3 + 6x^2 + 9x - 2x^2 - 12x - 18$

$P(x) = x^3 + 4x^2 - 3x - 18$ $\quad$ $a = 1 \quad b = 4 \quad c = -3 \quad d = -18$

c) A new function is formed by changing the signs of each of the values of a, b, c, and d. Describe how the graph of the new function compares to the graph of $P(x)$.

The graph of the new function is a reflection in the x-axis of the graph of P(x).

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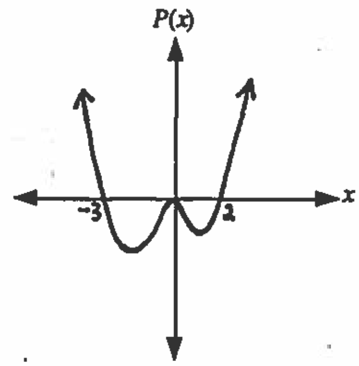
7. A polynomial function has the equation
$P(x) = x^2(x-2)(x+3)$.

a) Make a rough sketch without using a graphing calculator. Verify using a graphing calculator.

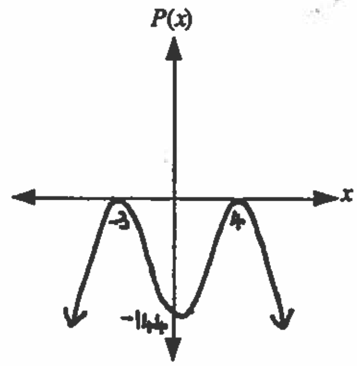


b) State the zeros, their multiplicities, and the y-intercept of $P(x)$.

zero	multiplicity	y-intercept
−3	1	
0	2	0
2	1	

8. A polynomial function has the equation
$P(x) = -(x-4)^2(x+3)^2$.

a) Make a rough sketch without using a graphing calculator. Verify using a graphing calculator.

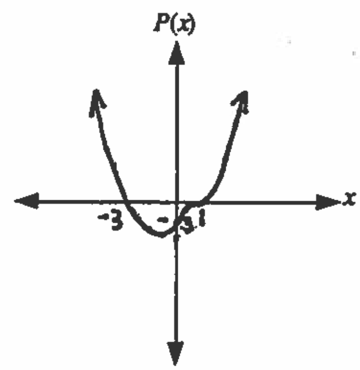


b) State the zeros, their multiplicities, and the y-intercept of $P(x)$.

zero	multiplicity	y-intercept
−3	2	$-(-4)^2(3)^2$
4	2	$= -144$

9. A polynomial function has the equation
$P(x) = (x-1)^3(x+3)$.

a) Make a rough sketch without using a graphing calculator. Verify using a graphing calculator.

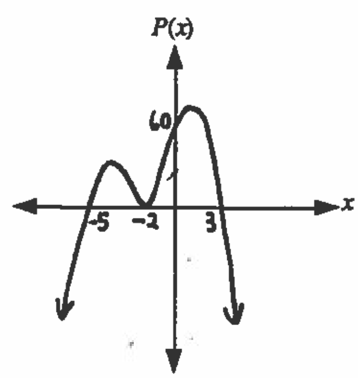


b) State the zeros, their multiplicities, and the y-intercept of $P(x)$.

zero	multiplicity	y-intercept
−3	1	$(-1)^3(3)$
1	3	$= -3$

10. A polynomial function has the equation
$P(x) = (x+2)^2(x+5)(3-x)$.

a) Make a rough sketch without using a graphing calculator. Verify using a graphing calculator.



b) State the zeros, their multiplicities, and the y-intercept of $P(x)$.

zero	multiplicity	y-intercept
−5	1	
−2	2	$(2)^2(5)(3)$
3	1	$= 60$

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11. Complete the following based on your observations from questions #1 - #10.

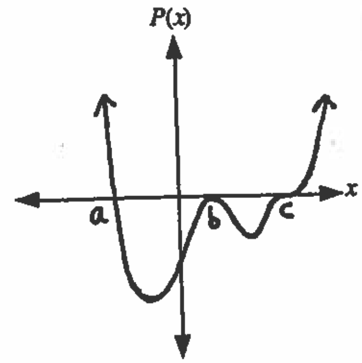
a) If a polynomial function has a zero of multiplicity 1 at $x = a$, then the graph of the function at $x = a$ __passes straight through the x-axis__.

b) If a polynomial function has a zero of multiplicity 2 at $x = b$, then the graph of the function at $x = b$ __is tangent to the x-axis__.

c) If a polynomial function has a zero of multiplicity 3 at $x = c$, then the graph of the function at $x = c$ __has a point of inflection__.

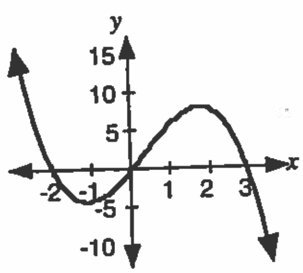
d) A polynomial function with a leading coefficient of 1 has three distinct zeros.
- a zero of multiplicity 1 at $x = a$
- a zero of multiplicity 2 at $x = b$
- a zero of multiplicity 3 at $x = c$

If $a < b < c$, make a rough sketch of a polynomial which satisfies these conditions.

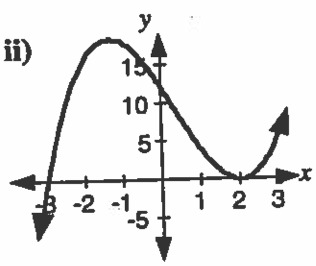


12. The graphs shown each represent a cubic polynomial function with equation $P(x) = ax^3 + bx^2 + cx + d$, where a is 1 or -1. The x-intercepts on the graphs are integers.

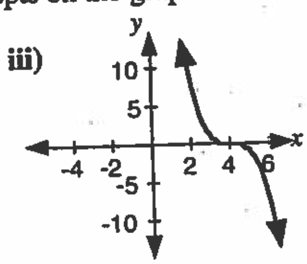
i)



ii)



iii)



In each case, write $P(x)$ in factored form and determine the values of a, b, c, and d.

i) leading coefficient is negative

zero	multiplicity
-2	1
0	1
3	1

$P(x) = -x(x+2)(x-3)$

$P(x) = -x(x^2 - x - 6)$
$= -x^3 + x^2 + 6x$

$a = -1 \quad b = 1 \quad c = 6 \quad d = 0$

ii) leading coefficient is positive

zero	multiplicity
-3	1
2	2

$P(x) = (x+3)(x-2)^2$

$P(x) = (x+3)(x^2 - 4x + 4)$
$= x^3 - 4x^2 + 4x + 3x^2 - 12x + 12$
$= x^3 - x^2 - 8x + 12$

$a = 1 \quad b = -1 \quad c = -8 \quad d = 12$

iii) leading coefficient is negative

zero	multiplicity
4	3

$P(x) = -(x-4)^3$

$P(x) = -(x-4)(x^2 - 8x + 16)$
$= -(x^3 - 8x^2 + 16x - 4x^2 + 32x - 64)$
$= -x^3 + 12x^2 - 48x + 64$

$a = -1 \quad b = 12 \quad c = -48 \quad d = 64$

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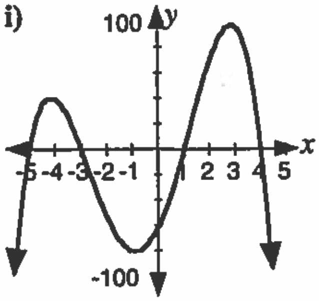
13. The graphs shown below each represent a quartic polynomial function with equation $P(x) = ax^4 + bx^3 + cx^2 + dx + e$, where a is 1 or -1. The zeros of the functions are integers.

In each case, write the equation of the polynomial function in factored form and determine the value of e.

i)

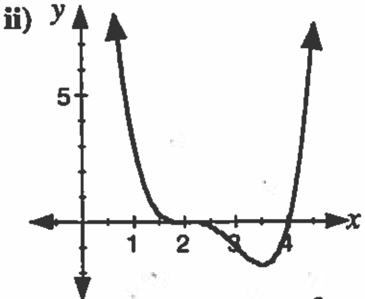


$P(x) = -(x+5)(x+3)(x-1)(x-4)$

$e = -(5)(3)(-1)(-4)$

$e = -60$

ii)

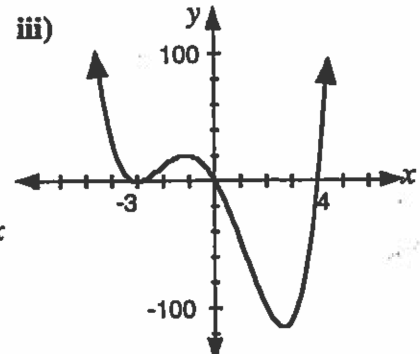


$P(x) = (x-4)(x-2)^3$

$e = (-4)(-2)^3$

$e = 32$

iii)

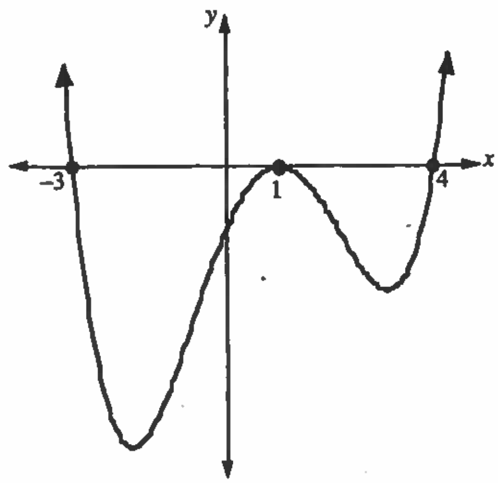


$P(x) = x(x+3)(x-4)$

$e = 0(3)(-4)$ $e = 0$

Use the following information to answer questions #14 and #15.

The partial graph of a fourth degree polynomial function $P(x)$ is shown. The leading coefficient is 1 and the x-intercepts of the graph are integers.



$P(x) = (x+3)(x-1)^2(x-4)$

$P(x) = (x-1)^2(x-4)(x+3)$

$c = 1$ $a = 1$ $b = 4$ $d = 3$

y-intercept $= (-1)^2(-4)(3)$

$= -12$

$m = 12$

Numerical Response

14. If the polynomial function is written in the form $P(x) = c(x-a)^2(x-b)(x+d)$, where a, b, c, and d are all positive integers, then the respective numerical values of a, b, c, d from left to right are _____.

(Record your answer in the numerical response box from left to right.)

1	4	1	3

15. The graph crosses the y-axis at $(0, -m)$. The value of m is _____.

(Record your answer in the numerical response box from left to right.)

1	2		

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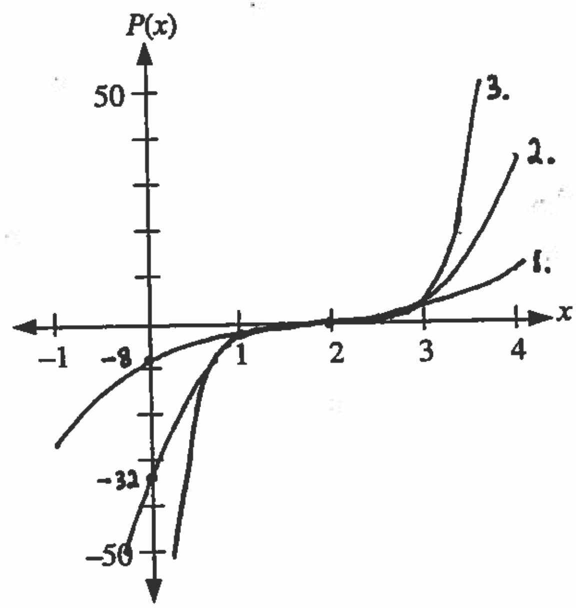
Polynomial Functions and Equations Lesson #9:
Investigating the Graphs of Polynomial Functions - Part Three

In this lesson we will investigate the graphs of polynomial functions which have zeros with multiplicities greater than three.

Investigating Odd Multiplicities

Graph the following functions on the grid showing the x- and y-intercepts.

1. $P(x) = (x-2)^3$ y int. −8
2. $P(x) = (x-2)^5$ y int. −32
3. $P(x) = (x-2)^7$ y int. −128



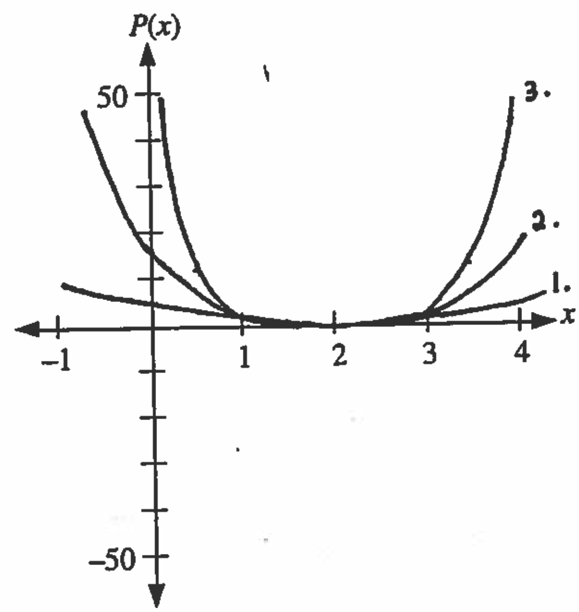
What happens as the multiplicity of the zero increases through the odd numbers?

All the graphs have a point of inflection at (2,0).

Investigating Even Multiplicities

Graph the following functions on the grid showing the x- and y-intercepts.

1. $P(x) = (x-2)^2$ y int. 4
2. $P(x) = (x-2)^4$ y int. 16
3. $P(x) = (x-2)^6$ y int. 64



What happens as the multiplicity of the zero increases through the even numbers?

All the graphs are tangent to the x-axis at (2,0).

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Even and Odd Multiplicities of a Zero

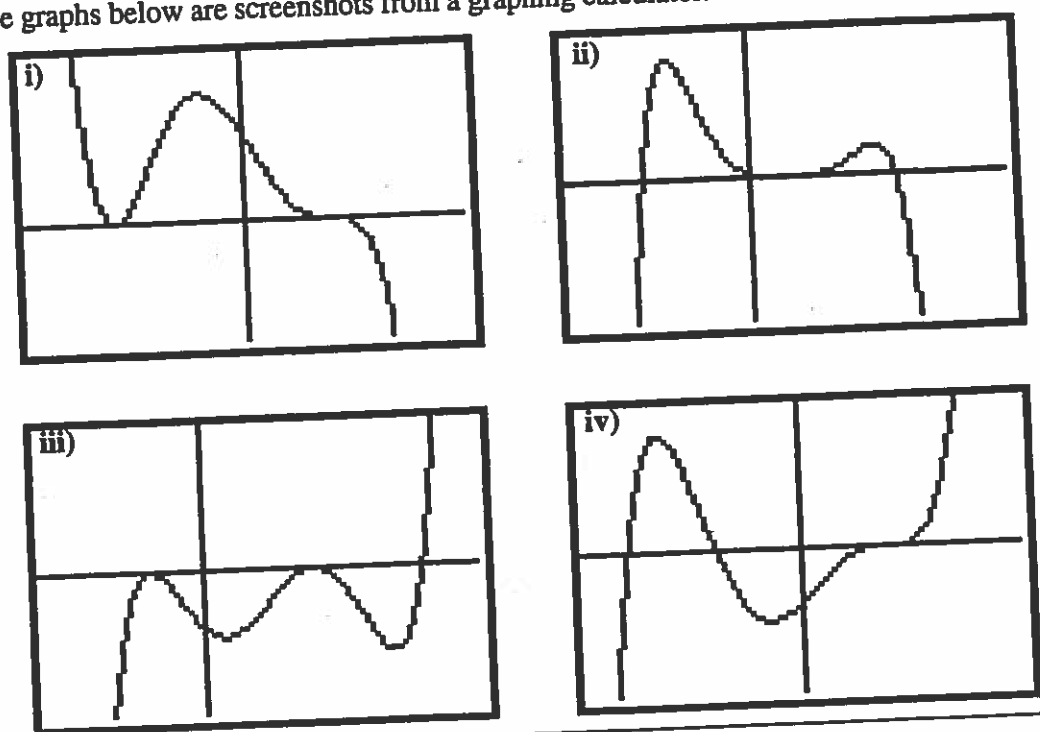
A real zero of <u>even</u> multiplicity (i.e. 2 or 4 or 6 or 8 or . . .) occurs where the graph of a polynomial function is <u>tangent</u> to the x-axis.

A real zero of <u>odd</u> multiplicity greater than 1 (i.e. 3 or 5 or 7 or . . .) occurs where the graph of a polynomial function has a <u>point of inflection</u> on the x-axis.

The sum of the multiplicities of the zeros of a polynomial function is equal to the degree of the polynomial function.

Use the following information to answer Class Example #1.

The graphs below are screenshots from a graphing calculator.



a) In each case state the number of distinct zeros and the possible multiplicities of each zero.

i) two distinct zeros
1 real zero of multiplicity 2 or 4 or 6 or ...
1 real zero of multiplicity 3 or 5 or 7 or ...

ii) three distinct zeros
2 real zeros of multiplicity 1
1 real zero of multiplicity 2 or 4 or 6 or ...

iii) three distinct zeros
1 real zero of multiplicity 1
2 real zeros of multiplicity 2 or 4 or 6 or ...

iv) three distinct zeros
2 real zeros of multiplicity 1
1 real zero of multiplicity 3 or 5 or 7 or ...

b) Which graph could represent a polynomial function of degree 8? ii)

c) In which of the graphs is the leading coefficient positive? iii), iv)

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Class Ex. #2

Without using a graphing calculator, make a rough sketch of the graph of

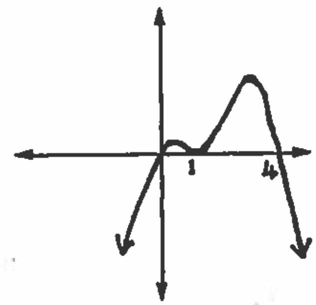
$$f(x) = -x(x-1)^4(x-4).$$

Verify using a graphing calculator.

leading coefficient is negative

zero at 0 and zero at 4 have multiplicity 1

zero at 1 has multiplicity 4



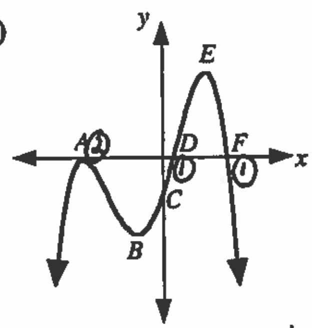
Class Ex. #3

The following graphs represent polynomial functions, $P(x)$, of lowest possible degree. In each case,

i) state the degree of the polynomial function;

ii) for $P(x) = 0$, state the points on the graph which represent real and equal roots.

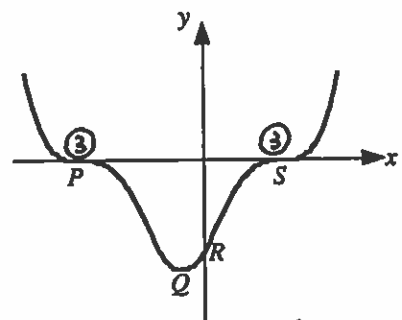
a)



i) 2 + 1 + 1 = 4

ii) A

b)



i) 3 + 3 = 6

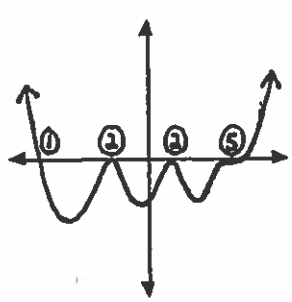
ii) P and S

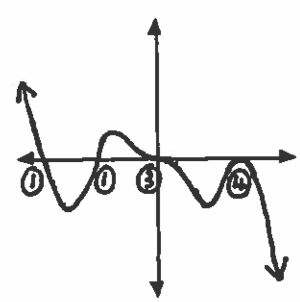
Class Ex. #4

a) On the grid, sketch a graph of a polynomial function satisfying the given conditions.

i)
- positive leading coefficient
- one real zero of multiplicity 1
- two real zeros of multiplicity 2
- one real zero of multiplicity 5

ii)
- negative leading coefficient
- two real zeros of multiplicity 1
- one real zero of multiplicity 3
- one real zero of multiplicity 4





b) State the degree of each polynomial function.

1 + 2 + 2 + 5 = 10

1 + 1 + 3 + 4 = 9

Complete Assignment Questions #1 - #5, #8, #9

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Extension: Investigating Non-Real Zeros

For a greater understanding of the graphs of polynomial functions, we investigate the concept of non-real zeros, and how they are represented on graphs.

Consider the fourth degree polynomial function, $P(x) = x^4 - 5x^2 + 8x - 12$.
The polynomial can be written in factored form as $P(x) = (x-2)(x+3)(x^2 - x + 2)$.

a) Using the factored form of $P(x)$, state two real zeros of multiplicity 1.

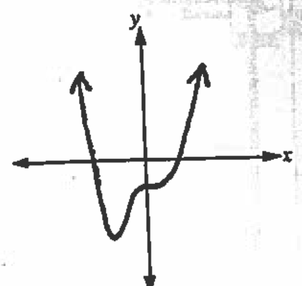
-3 and 2

b) Solve the equation $x^2 - x + 2 = 0$ using the quadratic formula to show that the other two zeros are **non-real.**

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(2)}}{2(1)} = \frac{1 \pm \sqrt{-7}}{2}$ both zeros are non-real

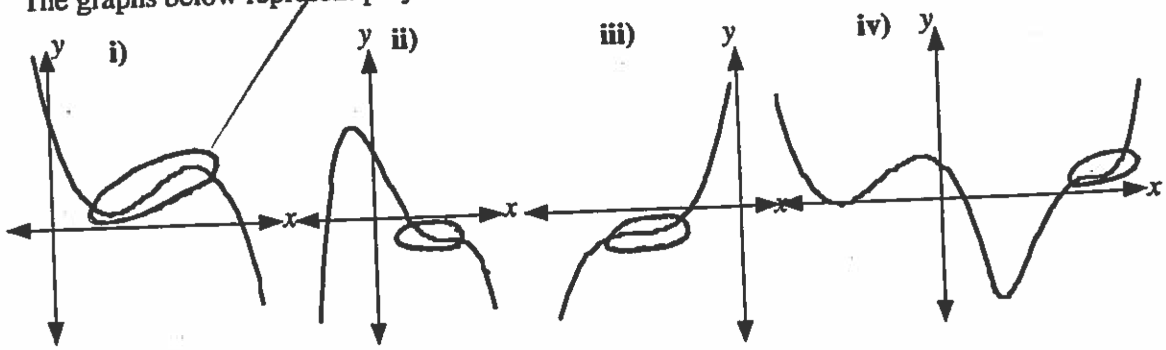
$a = 1 \quad b = -1 \quad c = 2$

c) To investigate how the non-real zeros $\frac{1-\sqrt{-7}}{2}$ and $\frac{1+\sqrt{-7}}{2}$ appear on a graph, use a graphing calculator to sketch $P(x) = x^4 - 5x^2 + 8x - 12$ on the grid provided. Use a graphing window x:[–8, 8, 1] y:[–40, 20, 10].



- If a real polynomial function has non-real zeros, then non-real zeros always occur in pairs which are conjugates of each other in the form $\frac{p \pm \sqrt{q}}{r}$, where $q < 0$.
- Non-real zeros are also known as **imaginary zeros** or **complex zeros.**

The graphs below represent polynomial functions which have non-real zeros.



a) Indicate the region on the graph which represents non-real zeros.
b) Write a statement which describes the number, type and multiplicity of the zeros.

i) 1 real zero of multiplicity 1 , 2 non-real zeros
ii) 2 real zeros of multiplicity 1 , 2 non-real zeros
iii) 1 real zero of multiplicity 1 , 2 non-real zeros
iv) 2 real zeros of multiplicity 1 , 1 real zero of multiplicity 2 or 4 or 6 or ... , 2 non-real zeros

Complete Assignment Questions #6 and #7

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Assignment

1. How does the concept **number of zeros** differ from the concept **multiplicity of zeros**?

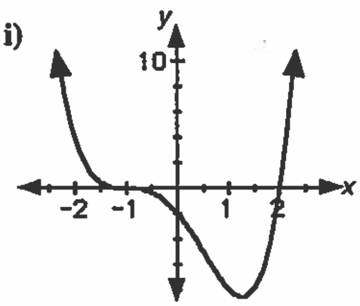
Number of zeros refers to how many distinct zeros the function has.

Multiplicity refers to the number of times a zero repeats.

2. Consider the graphs below.

a) In each case state the number of distinct zeros and the possible multiplicities of each zero.

i)

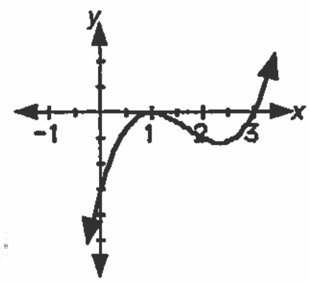


two distinct zeros

the zero −1 has multiplicity 3 or 5 or 7 or...

the zero 2 has multiplicity 1

ii)

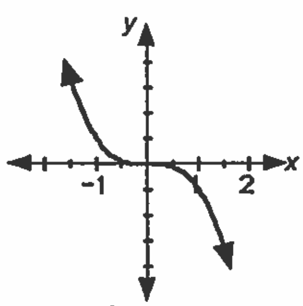


two distinct zeros

the zero 1 has multiplicity 2 or 4 or 6 or ...

the zero 3 has multiplicity 1

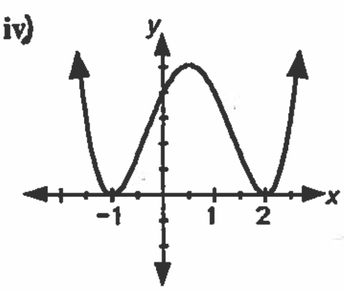
iii)



one distinct zero

the zero 0 has multiplicity 3 or 5 or 7 or...

iv)



two distinct zeros

both the zeros −1 and 2 have multiplicity 2 or 4 or 6 or ...

b) Which graph(s) could represent a seventh-degree polynomial function? ii), iii)

c) Which graph(s) could **not** represent a polynomial function of degree 10? ii), iii), iv)

In graph iv) the zeros are of equal multiplicity which cannot be 5.

d) In which of the graphs is the leading coefficient negative? iii)

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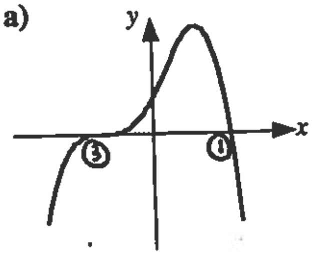
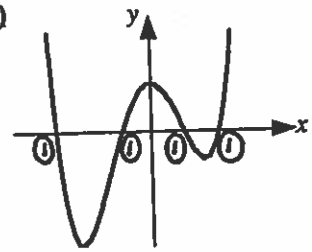
3. The following graphs represent functions of lowest possible degree. State the degree in each case.

a)



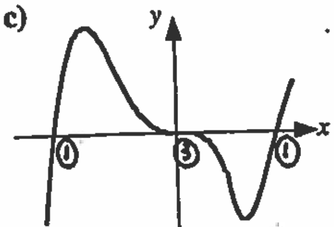
$3 + 1 = \underline{4}$

b)



$1 + 1 + 1 + 1 = \underline{4}$

c)



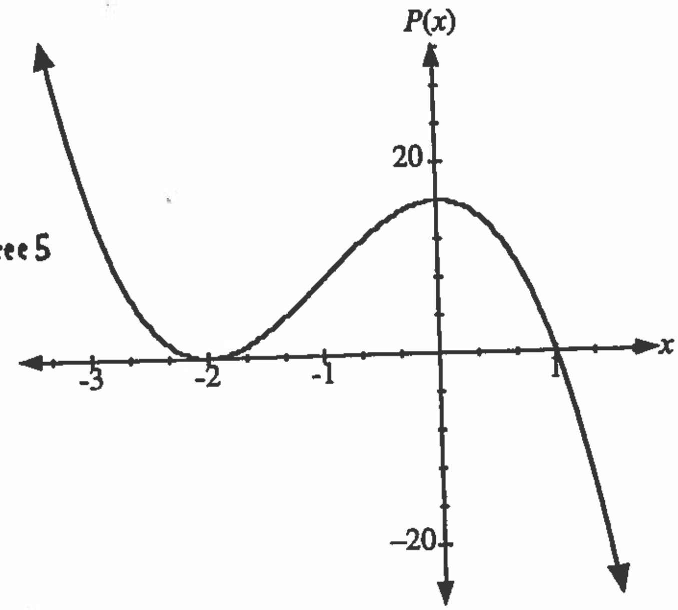
$1 + 3 + 1 = \underline{5}$

4. The graph represents a polynomial function $P(x)$ of degree 5.

Write the equation of $P(x)$ in factored form if the leading coefficient is -1.

zero -2 has multiplicity 4
zero 1 has multiplicity 1
} degree 5

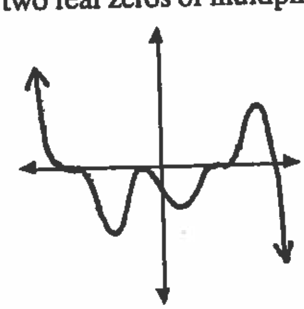
$\underline{P(x) = -(x-1)(x+2)^4}$



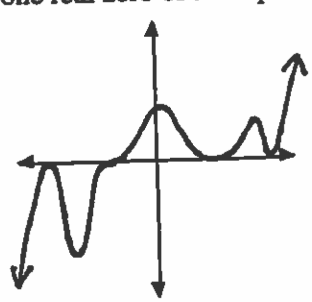
5. a) On the grid, sketch a graph of a polynomial function satisfying the given conditions.

i)
- negative leading coefficient
- one real zero of multiplicity 1
- one real zero of multiplicity 2
- two real zeros of multiplicity 3

ii)
- positive leading coefficient
- two real zeros of multiplicity 2
- one real zero of multiplicity 3
- one real zero of multiplicity 6



many answers are possible.



b) State the degree of each polynomial function.

$3 + 2 + 3 + 1 = \underline{9}$

$2 + 3 + 6 + 2 = \underline{13}$

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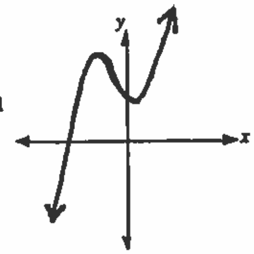
6. The cubic function $f(x) = x^3 - 12x + 65$ has one real zero of multiplicity 1 and two non-real zeros.



   a) Use a graphing calculator to graph the function and make a sketch of the graph on the grid.

   b) Use synthetic division to determine the real zero of the function.

$$\begin{array}{r|rrrr} -5 & 1 & 0 & -12 & 65 \\ & & -5 & 25 & -65 \\ \hline & 1 & -5 & 13 & 0 \end{array}$$

The real zero is $\underline{\underline{-5}}$

$f(x) = (x+5)(x^2-5x+13)$

   c) Determine the two non-real zeros of the function.

solve $x^2 - 5x + 13 = 0$

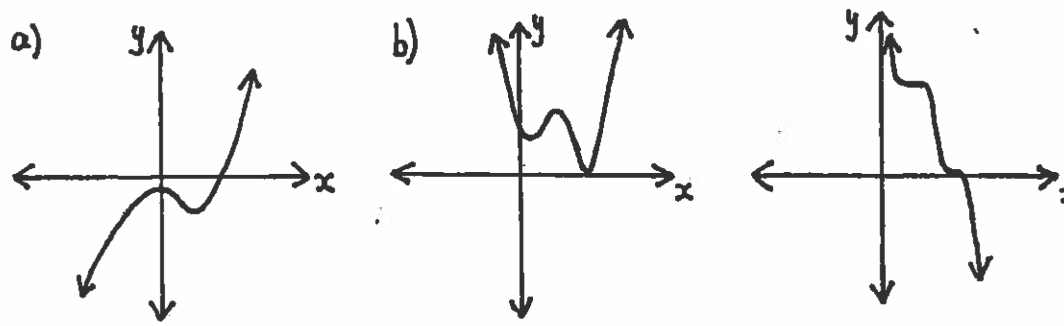
$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(13)}}{2(1)} = \frac{5 \pm \sqrt{-27}}{2} = \underline{\underline{\frac{5 \pm 3\sqrt{-3}}{2}}}$$

7. In each case, sketch the graph of the polynomial function using a graphing calculator window of x:[–8, 8, 1] and y:[–50, 50, 10] and write a statement which describes the number, type, and multiplicity of the zeros.

   a) $P(x) = x^3 - 6x^2 + 6x - 5$
   One real zero of multiplicity 1.
   Two non-real zeros, each of multiplicity 1.

   b) $Q(x) = x^4 - 11x^3 + 36x^2 - 35x + 25$
   One real zero of multiplicity 2.
   Two non-real zeros, each of multiplicity 1.

   c) $R(x) = -x^5 + 13x^4 - 61x^3 + 124x^2 - 112x + 64$
   One real zero of multiplicity 3.
   Two non-real zeros, each of multiplicity 1.



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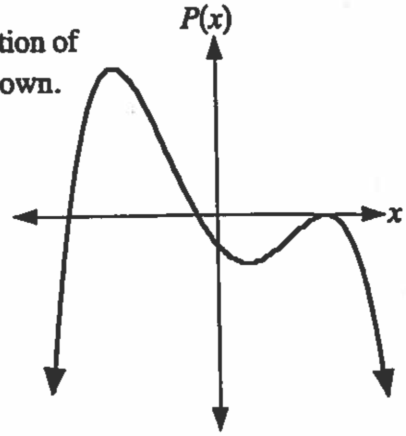
Use the following information to answer the next two questions.

The graph of a fourth degree polynomial function of the form $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ is shown.



Multiple Choice

8. The values a and e must satisfy

A. $a>0, e<0$ **B.** $a<0, e>0$ **C.** $a>0, e>0$ **D.** $a<0, e<0$

leading coefficient is negative so $a<0$ y-intercept is negative so $e<0$

9. If $P(x) = 0$ has exactly three different solutions, then which one of the following statements about the roots of $P(x) = 0$ is true?

A. Two roots are real, equal and negative, and two roots are real, not equal and positive.
B. Two roots are real, equal and positive and two roots are real, not equal and negative.
C. Two roots are real and negative, and two roots are not real.
D. Two roots are real and positive, and two roots are not real.

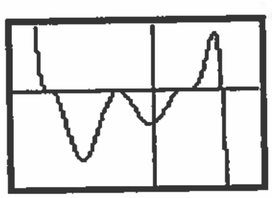
Answer Key

1. - number of zeros refers to how many distinct zeros the function has
   - multiplicity refers to the number of times a zero repeats

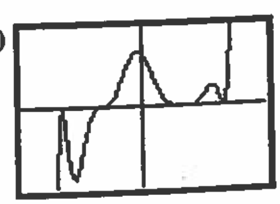
2. a) i) two zeros, the zero -1 has a multiplicity of 3 or 5 or 7 ..., the zero 2 has a multiplicity of 1.
   ii) two zeros, the zero 1 has a multiplicity of 2 or 4 or 6 ..., the zero 3 has a multiplicity of 1.
   iii) one zero, the zero 0 has a multiplicity of 3 or 5 or 7 ... .
   iv) two zeros, both the zeros -1 and 2 have a multiplicity of 2 or 4 or 6 ...,
   b) ii), iii) c) ii), iii), iv) d) iii)

3. a) 4 b) 4 c) 5 4. $P(x) = -(x+2)^4(x-1)$

5. a) Answers may vary → → → → → → i)



ii)



   b) i) 9 ii) 13

6. b) -5 c) $\dfrac{5-3\sqrt{-3}}{2}$ and $\dfrac{5+3\sqrt{-3}}{2}$

7. a) One real zero of multiplicity 1, and two non-real zeros, each of multiplicity 1.
   b) One real zero of multiplicity 2, and two non-real zeros, each of multiplicity 1.
   c) One real zero of multiplicity 3, and two non-real zeros, each of multiplicity 1.

8. D 9. B

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Polynomial Functions and Equations Lesson #10: Polynomial Functions with a Leading Coefficient other than ±1

Review

In lesson 7 we introduced the Unique Factorization Theorem, which states that every polynomial function of degree $n \geq 1$ can be written as the product of a leading coefficient, c, and n linear factors to get

$$P(x) = c(x - a_1)(x - a_2)(x - a_3)\ldots(x - a_n)$$

This theorem implies two ***important*** points for polynomial functions of degree $n \geq 1$:

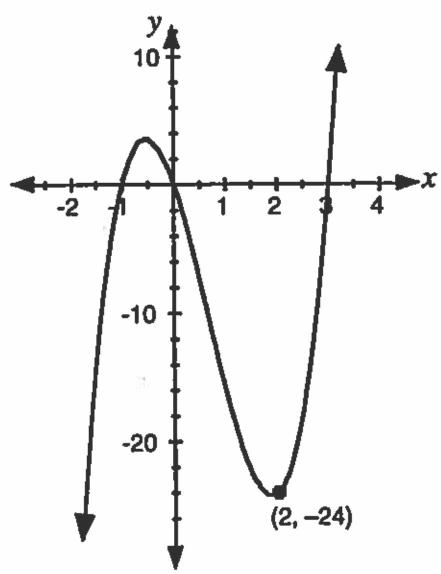
Point #1: Every polynomial function can be written as a product of its factors and a leading coefficient.

Point #2: Every polynomial function has the same number of factors as its degree. The factors may be real or complex, and may be repeated.

In lessons 7 - 9 we have considered only polynomial functions where the leading coefficient, c, was either 1 or –1.

In this lesson we will consider polynomial functions where the leading coefficient, c, can be any real number.

The graph of a third degree polynomial function, $P(x)$, is shown. The graph has integral x-intercepts and passes through the point $(2, -24)$.



a) Explain why the equation of the polynomial function can be written in the form $P(x) = cx(x + 1)(x - 3)$.

There are three distinct zeros (0, −1 and 3) each of multiplicity 1.

b) Use the point $(2, -24)$ to determine the value of c, and hence write the polynomial in expanded form.

$P(x) = c\,x(x+1)(x-3)$

$-24 = c(2)(2+1)(2-3)$

$-24 = -6c$

$c = 4$

$P(x) = 4x(x+1)(x-3) = 4x(x^2-2x-3) = \underline{\underline{4x^3-8x^2-12x}}$

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Class Ex. #2

The graph represents a polynomial function of lowest possible degree. The intercepts are integers.

Determine the equation of the polynomial function in factored form.

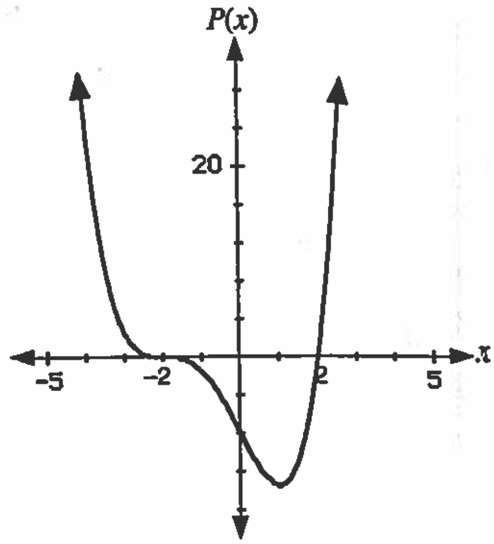
$P(x) = c(x-2)(x+2)^3$

y-intercept $= -8$

$-8 = c(0-2)(0+2)^3$

$-8 = -16c \qquad c = \frac{1}{2}$

$\underline{\underline{P(x) = \frac{1}{2}(x-2)(x+2)^3}}$



Class Ex. #3

Determine the equation, in factored form, of a fourth degree polynomial function which passes through $(1, -12)$ and is tangent to the x-axis at $(2, 0)$ and at $(-3, 0)$.

If the polynomial has degree 4 then both the zeros 2 and −3 have multiplicity 2.

$P(x) = c(x-2)^2(x+3)^2$

$-12 = c(1-2)^2(1+3)^2$

$-12 = c(1)(16)$

$-12 = 16c$

$c = \frac{-12}{16} = -\frac{3}{4}$

$\underline{\underline{P(x) = -\frac{3}{4}(x-2)^2(x+3)^2}}$

Complete Assignment Questions #1 - #5

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Class Ex. #4

A fourth degree polynomial, $P(x)$, passes through the point (1, 2) and has zeros $-1, 0, \frac{3}{2}$, and 2.

Determine the equation of $P(x)$ in factored form using only integral linear factors.

Each of the zeros has multiplicity 1.

$$P(x) = c\,x(x+1)(2x-3)(x-2)$$

$$2 = c(1)(1+1)(2(1)-3)(1-2)$$

$$2 = c(1)(2)(-1)(-1)$$

$$2 = 2c$$

$$c = 1$$

$$\underline{\underline{P(x) = x(x+1)(2x-3)(x-2)}}$$

$x = \frac{3}{2}$

$2x = 3$

$2x - 3 = 0$

$2x-3$ is a factor

Class Ex. #5

A polynomial equation has the following three roots.

- -2 is a root with a multiplicity of 1
- $\frac{1}{3}$ is a root with a multiplicity of 2
- 1 is a root with a multiplicity of 3

The graph of the corresponding polynomial function has a y-intercept of $\frac{2}{3}$.

Determine the equation of the polynomial function in factored form using integral factors.

$$P(x) = c(x+2)(3x-1)^2(x-1)^3$$

$$\frac{2}{3} = c(0+2)(3(0)-1)^2(0-1)^3$$

$$\frac{2}{3} = c(2)(-1)^2(-1)^3$$

$$\frac{2}{3} = -2c$$

$$c = -\frac{1}{3}$$

$$\underline{\underline{P(x) = -\frac{1}{3}(x+2)(3x-1)^2(x-1)^3}}$$

$x = \frac{1}{3}$

$3x = 1$

$3x - 1 = 0$

$3x-1$ is a factor

Complete Assignment Questions #6 - #14

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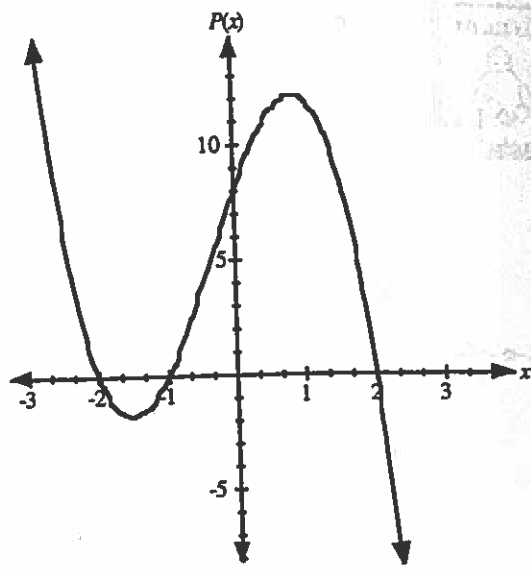
Assignment

1. The graph of the polynomial function shown has integral intercepts.

   Determine the equation of the function in factored form.



$P(x) = c(x+2)(x+1)(x-2)$

$(0, 8) \Rightarrow P(0) = 8$

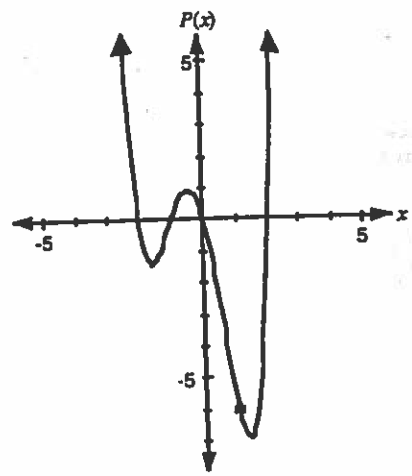
$8 = c(0+2)(0+1)(0-2)$

$8 = -4c \qquad c = -2$

$\underline{\underline{P(x) = -2(x+2)(x+1)(x-2)}}$

2. The graph passes through the point (1, –6) and has integral x-intercepts.

   Determine the equation, in factored form, of the polynomial function, $P(x)$, represented by the graph.



$P(x) = cx(x+2)(x+1)(x-2)$

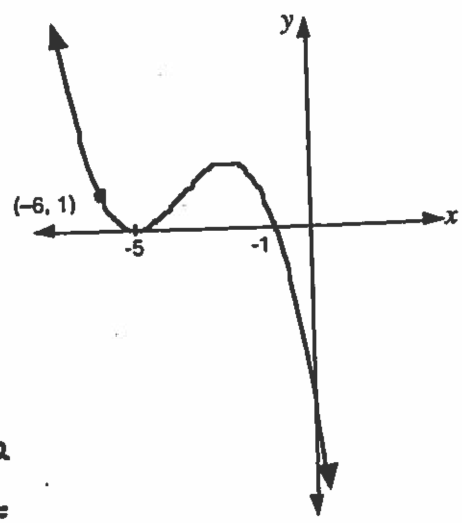
$-6 = c(1)(1+2)(1+1)(1-2)$

$-6 = -6c$

$c = 1$

$\underline{\underline{P(x) = x(x+2)(x+1)(x-2)}}$

3. The graph of a third degree polynomial function is shown. The graph passes through the point (–6, 1). If the polynomial function has zeros –5 and –1, determine



   a) the equation of the function in factored form;

zero 5 has multiplicity 2 } degree = 3
zero –1 has multiplicity 1 }

$P(x) = c(x+5)^2(x+1)$

$1 = c(-6+5)^2(-6+1)$

$1 = -5c$

$c = -\frac{1}{5}$

$\underline{\underline{P(x) = -\frac{1}{5}(x+1)(x+5)^2}}$

   b) the y-intercept of the graph.

$P(0) = -\frac{1}{5}(0+1)(0+5)^2 = -5 \qquad$ y-intercept $= \underline{\underline{-5}}$

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4. Find the equation of a quartic function whose graph has a point of inflection at the origin and passes through (4, 0) and (–1, 10).

$\left.\begin{array}{l}\text{zero } 0 \text{ has multiplicity } 3\\ \text{zero } 4 \text{ has multiplicity } 1\end{array}\right\}$ degree 4

$P(x) = cx^3(x-4)$

$10 = c(-1)^3(-1-4)$

$10 = 5c \qquad c = 2$

$$\underline{\underline{P(x) = 2x^3(x-4)}}$$

5. The design of a route for a cross country ski course was drawn on a Cartesian plane. The route is tangent to the x-axis at (1, 0) and (–3, 0). It crosses the x-axis at (–5, 0) and also passes through the point (–2, 9). Determine the equation of the fifth degree polynomial function that will meet these conditions.

$\left.\begin{array}{l}\text{zeros } 1 \text{ and } -3 \text{ have multiplicity } 2\\ \text{zero } -5 \text{ has multiplicity } 1\end{array}\right\}$ degree 5

$P(x) = c(x+5)(x-1)^2(x+3)^2$

$9 = c(-2+5)(-2-1)^2(-2+3)^2$

$9 = c(3)(9)(1)$

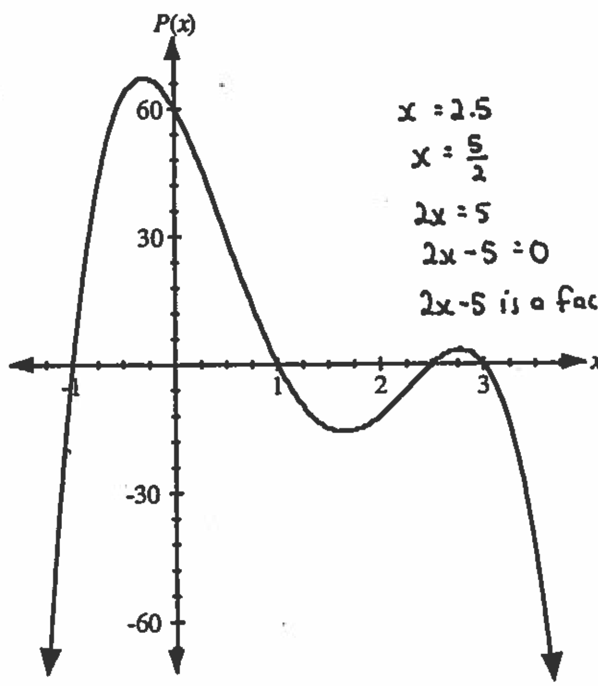
$9 = 27c$

$c = \frac{1}{3}$

$$\underline{\underline{P(x) = \tfrac{1}{3}(x+5)(x-1)^2(x+3)^2}}$$

6. The graph shown has x-intercepts of –1, 1, 2.5, and 3 and a y-intercept of 60.

Determine the equation of the graph in factored form using integral factors.



$x = 2.5$

$x = \frac{5}{2}$

$2x = 5$

$2x - 5 = 0$

$2x-5$ is a factor

$P(x) = c(x+1)(x-1)(2x-5)(x-3)$

$60 = c(0+1)(0-1)(2(0)-5)(0-3)$

$60 = c(1)(-1)(-5)(-3)$

$60 = -15c$

$c = -4$

$$\underline{\underline{P(x) = -4(x+1)(x-1)(2x-5)(x-3)}}$$

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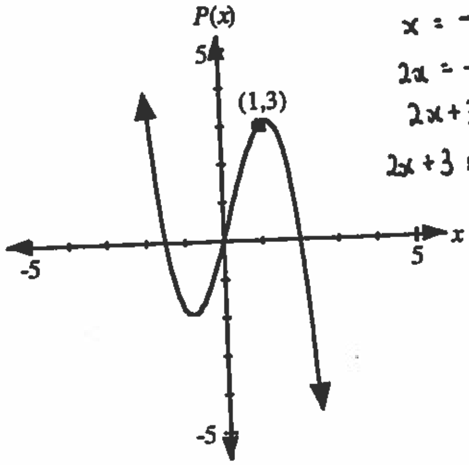
7. The graph below has x-intercepts $-\frac{3}{2}$, 0, and 2 and passes through the point (1, 3). Determine the equation of the graph using integral factors.



$x = -\frac{3}{2}$

$2x = -3$

$2x+3 = 0$

$2x+3$ is a factor

$P(x) = c(2x+3)(x)(x-2)$

$3 = c(2(1)+3)(1)(1-2)$

$3 = c(5)(1)(-1)$

$3 = -5c$

$c = -\frac{3}{5}$

$$\underline{\underline{P(x) = -\frac{3}{5}x(2x+3)(x-2)}}$$

8. Determine the equation of a fifth degree polynomial function whose graph has a point of inflection at (3, 0), is tangent to the x-axis at $\left(\frac{1}{2}, 0\right)$, and passes through (2, 1).

zero 3 has multiplicity 3
zero $\frac{1}{2}$ has multiplicity 2
} degree 5

$x = \frac{1}{2}$ $\quad 2x = 1$ $\quad 2x-1 = 0$

$2x-1$ is a factor

$P(x) = c(x-3)^3(2x-1)^2$

$1 = c(2-3)^3(2(2)-1)^2$

$1 = c(-1)(9)$

$1 = -9c$

$c = -\frac{1}{9}$

$$\underline{\underline{P(x) = -\frac{1}{9}(x-3)^3(2x-1)^2}}$$

Multiple Choice **9.** If the zeros of a polynomial are -1, $\frac{1}{2}$ and $\frac{2}{3}$, then the polynomial could be

A. $12x^3 - 2x^2 + 10x - 4$
B. $6x^3 + x^2 - 5x + 2$
C. $30x^3 - 5x^2 - 25x + 10$
D. $18x^3 + 3x^2 - 15x - 6$

$P(x) = c(x+1)(2x-1)(3x-2)$

$= c(x+1)(6x^2-7x+2)$

$= c(6x^3-7x^2+2x+6x^2-7x+2)$

$= c(6x^3-x^2-5x+2)$

The only multiple of $6x^3-x^2-5x+2$ is D.

$3(6x^3-x^2-5x+2) = 18x^3-3x^2-15x+6$

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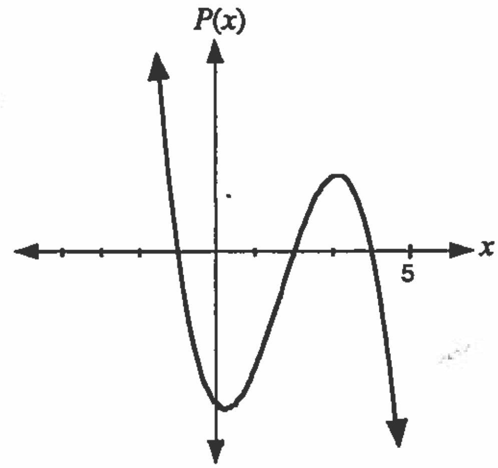
10. $P(x) = -3x^3 + bx^2 + cx + d$ is an integral polynomial function with zeros 2, –1, and 4. A sketch of $y = P(x)$ is shown.

At which of the following points does the graph of $P(x)$ cross the y-axis?

A. (0, –8) **B.** (0, –15)

C. (0, –16) **D.** (0, –24)



$P(x) = c(x+1)(x-2)(x-4)$

$= c(x+1)(x^2-6x+8)$

$= c(x^3-6x^2+8x+x^2-6x+8)$

$= c(x^3-5x^2+2x+8)$

Since the coefficient of x^3 is -3 $c = -3$

$y_{int} = P(0) = -3(8) = -24$

11. The graph of a fourth degree polynomial function has x-intercepts –2, –1, 0, and 1. If the graph passes through the point (–3, –48), then the coefficient of the third degree term of $P(x)$ is

A. –4
B. –2
C. –1
D. 2

$P(x) = cx(x+2)(x+1)(x-1)$

$-48 = c(-3)(-3+2)(-3+1)(-3-1)$

$-48 = c(-3)(-1)(-2)(-4)$

$-48 = 24c$

$c = -2$

$P(x) = -2x(x+2)(x+1)(x-1)$

$P(x) = -2x(x+2)(x^2-1)$

$= -2x(x^3+2x^2-x-2)$

$= -2x^4-4x^3+2x^2+4x$

coefficient of x^3 is -4

12. A third degree polynomial, $f(x)$, has three distinct zeros, –3, –1, and 2. If a new polynomial, $g(x)$, is found by multiplying $f(x)$ by $(x + 1)$, then which of the following statements is true?

A. The x-intercepts of the graph of $y = g(x)$ will be –4, –2, and 1.

B. The x-intercepts of the graph of $y = g(x)$ will be –2, 0, and 3.

C. The y-intercept of the graph of $y = g(x)$ will be the negative of the y-intercept of the graph of $y = f(x)$.

D. The x and y-intercepts of the graph of $y = g(x)$ are the same as the x and y intercepts of the graph of $y = f(x)$.

$f(x) = c(x+3)(x+1)(x-2)$ $x_{int} = -3, -1, 2$ $y_{int} = c(3)(1)(-2) = -6c$

$g(x) = c(x+3)(x+1)^2(x-2)$ $x_{int} = -3, -1, 2$ $y_{int} = c(3)(1)^2(-2) = -6c$

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